coordinate system changes the zero of the ψ scale by 90° and reverses all the phases determined by this method for $0,0,(4n+2)$ structure factors. This is not an error. In space group $P2_13$ the c axis lacks fourfold symmetry. The analytical description of equivalent positions used to derive (1) is

$$x, x, x;\quad \tfrac{1}{2}+x, \tfrac{1}{2}-x, -x;$$
$$-x, \tfrac{1}{2}+x, \tfrac{1}{2}-x;\quad \tfrac{1}{2}-x, -x, \tfrac{1}{2}+x.$$

The 90° rotation of cooordinates requires that the origin be moved (*e.g.* by $\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$) and that a new value be chosen for x in order to describe the structure with this list of equivalent positions. This shift of origin changes these phases by 180°.

This example is a special case which is simplified by constraints on the scattering tensor from the threefold symmetry of the special positions and by interference of scattering from two pairs of atoms. More complicated relationships occur for general positions or for some other space groups (Dmitrienko, 1983, 1984). It may not be easy to exploit these effects in every case, but these kinds of data are rich in phase information. We hope that this report will stimulate more study of them.

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Polyhedra of Three Quasilattices Associated with the Icosahedral Group

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Abstract

A systematic approach is presented for the construction of families of polytopes associated with a given group G and subgroup H. This procedure is applied to the icosahedral group and its three dihedral subgroups yielding three families of polyhedra in $\mathbb{E}^3$. These families form the cells of three types of quasilattices associated with the icosahedral group, and hence are candidates for modelling quasicrystal structures. Some of the polyhedra are illustrated.

1. Introduction

In classical crystallography the icosahedral group $A(5)$ plays no role as it cannot occur as a subgroup of a crystallographic space group in $\mathbb{E}^3$ except in relation to approximate local site symmetry (*cf.* Hahn & Klapper, 1983; § 10.4). The discovery of diffraction patterns with icosahedral point symmetry in 4:1 Al–Mn by Shechtman, Blech, Gratias & Cahn (1984) shows that this group cannot be excluded from crystallography. A type of non-periodic space filling in $\mathbb{E}^3$ obtained as a projection from $\mathbb{E}^{12}$ or $\mathbb{E}^6$ was associated with $A(5)$ by Kramer & Neri (1984). The cells of this space filling were shown by Kramer (1985) to form a family of polyhedra in $\mathbb{E}^3$ bounded by pairs of parallel rhombus faces. In the present paper we implement a systematic approach to the construction of families of polytopes associated with a given group G and subgroup H.

When a group G acts as a transformation group on a set X, this set can be partitioned into orbits $Gx', Gx'', \ldots, x', x'' \in X$. Each orbit is a homogeneous space under G and can be characterized up to conju-

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gation by a stability group $H < G$. The elements of a fixed orbit can be brought into one-to-one correspondence with the elements of the coset space G/H. We refer to Hermann (1966; Ch. 2) for a short exposition of these concepts. In standard crystallography, the concept of orbits is used, for example, in the space-group analysis of Wyckoff positions (*cf.* Wondratschek, 1983; § 8.3.2), and it appears in the theory of space groups acting on Euclidean space (Brown, Bülow, Neubüser, Wondratschek & Zassenhaus, 1978). In linear representation theory, a pair of groups $G > H$ gives rise to the concepts of subduction and induction. Subduction theory deals with a representation of G restricted to H, induction theory builds from any representation of H a representation of G. The general concepts of subduction, induction, and reciprocity, which links them together, are reviewed by Coleman (1968).

We shall make use of these concepts in the present paper. In Part A, we treat in very general terms three topics which are linked together in any particular application. First, the action of a group G on various partitions of the set of cosets from G/H is subject to an orbit analysis. Second, certain results from the theory of induced representations with respect to G and H and projection operators are given. The latter is to be compared with §13.7 of Coxeter (1963). Our presentation follows that of Haase & Butler (1984) and Haase & Dirl (1987). Third, systems of p vectors in $\mathbb{E}^n (p \geq n)$ are associated with corresponding polytopes. The method of constructing these polytopes for $\mathbb{E}^3$, *i.e.* polyhedra, follows Coxeter (1963; § 2.8) and is a more direct method than the dual grid method used by Kramer (1985). With respect to the systems of vectors called stars, we also refer to Seidel (1976).

We outline the connection between these topics. In special cases the basis vectors of the induced representation space form under projection a star in $\mathbb{E}^3$. The vectors of this star lie along the symmetry axes of some regular polyhedra with symmetry group G, each axis displaying the symmetry H. The orbit analysis of G/H provides a classification of all polyhedra obtained from the star. It is these polyhedra which form space fillings of $\mathbb{E}^3$. Depending on the groups G and H, the space filling can be periodic or non-periodic. In the latter case, the structure is called a quasilattice. In this context the concept of determining cells for quasilattices from the vertices of polyhedra was prescribed by Lalvani (1986). The number of vertex arrangements of polyhedra, which determines the number of vertex stars, can be derived from the method of vertex combination suggested by Lalvani (1981).

In Part B, the analysis is applied to the icosahedral group $A(5)$ and its dihedral subgroups $D(m)$, $m = 5, 3, 2$. The non-trivial one-dimensional real orthogonal representations of these subgroups induce representations of dimension $n = 6, 10, 15$ of $A(5)$. These induced representations contain upon subduction the three-dimensional irreducible representation $[31_+^2]$ of $A(5)$. Thus, the basis vectors of the three induced representations in $\mathbb{E}^n$ form three stars which are projected into $\mathbb{E}^3$. The n projected vectors of the stars ($n = 6, 10, 15$) point to the vertices of the icosahedron, dodecahedron and icosidodecahedron respectively, and yield three families of polyhedra in $\mathbb{E}^3$. From the $n = 6$ analysis of Kramer & Neri (1984), these polyhedra form the cells for three types of quasilattices associated with the icosahedral group. These quasi-lattices are possible candidates for the modelling of quasicrystals such as the 4:1 Al–Mn alloy discovered by Shechtman *et al.* (1984).

Since the number of polyhedra is large ($n = 10$ has 44 polyhedra and $n = 15$ has 680) we present figures for only some of the polyhedra. We include the largest of each family. These are the rhombic triacontahedron, the rhombic enneahedron, and Kepler's truncated icosidodecahedron, which naturally have the full icosahedral group symmetry. The other polyhedra may be viewed as being obtained by removing one or more sets of parallel edges from the largest polyhedra.

Part A: Algebraic and geometric structures

We describe several algebraic and geometric structures which will form the basis of the work that is to follow in later sections. We begin with those algebraic structures concerning a group G and a subgroup H. In particular, the induced representations from H into G will be important as this leads to certain Euclidean spaces. General projection methods applied to the induced representation space onto irreducible representations of G are given. Finally, the general methods of constructing polyhedra from a set of p vectors in $\mathbb{E}^n (p \geq n)$ are described.

2. Group–subgroup coset analysis

Consider a group G and a subgroup H. For our purpose we assume both are finite groups. The division of G into left cosets of H partitions the group elements. We write

$$G = \bigcup_{i=1}^{n} q_i H, \quad i = 1, \ldots, n = |G|/|H|. \tag{1}$$

Each element of G can be expressed as $g = qh$ where $g \in G$, $h \in H$ and q is the coset representative. The set of cosets forms the coset space G/H,

$$G/H = \{q_1 H, \ldots, q_n H\}. \tag{2}$$

For a given coset its representative is not unique, but a fixed representative fixes a coset. This allows us to denote the coset space G/H by the set of representatives $\{q_1, \ldots, q_n\}$.

A group action $G: G/H \to G/H$ can be defined:

$$\text{for every } g \in G \quad gq_i \to q_{\alpha(i)} \quad \text{such that } q_{\alpha(i)}^{-1} g q_i \in H. \tag{3}$$

The mapping permutes the coset representatives. In this manner the group G is embedded into the symmetric group $S(n)$ of all permutations of the coset representatives. The group action determines a homomorphism from the abstract group into a transformation group on the coset space. In the present notation we suppress the distinction.

Now we introduce for fixed p, $1 \le p \le n$, the set X_p^n of all partitions of G/H into two subsets of p and $n-p$ elements respectively:

$$X_p^n = \{(q_{\alpha(1)}, \ldots, q_{\alpha(p)})(q_{\alpha(p+1)}, \ldots, q_{\alpha(n)}) \,|\, \alpha(i) \in 1, \ldots, n\}. \tag{4}$$

The group action $G: G/H \to G/H$ generates a group action $G: X_p^n \to X_p^n$, and, furthermore, X_p^n can be classified into orbits under G. This is done by choosing a representative partition from X_p^n and determining its stability group S_p^n with the property that it transforms the representative partition into itself. Clearly S_p^n has the form $G \cap [S(p) \times S(n-p)]$ where the symmetric groups act on $\alpha(1) \ldots \alpha(p)$ and $\alpha(p+1) \ldots \alpha(n)$ respectively. Application of the coset representatives from G/S_p^n then generates an orbit of $|G|/|S_p^n|\,(p, n-p)$ partitions within X_p^n. For fixed n and p, we have the sum rule

$$\sum_b |G|/|S_p^n(b)| = \binom{n}{p}, \tag{5}$$

where the different orbits are denoted by an additional label b. There exists a complementarity between the orbits of X_p^n and X_{n-p}^n. Also their stability groups must be isomorphic, $S_{n-p}^n \simeq S_p^n$.

The orbit analysis is well determined once a presentation of the group elements in terms of generators can be given and a set of coset representatives is chosen.

3. Induced representation theory

The next structure to be outlined is the construction of the induced representation space of G from an *ir*reducible *rep*resentation (irrep) space of H. The induced representation $\eta(H)\uparrow G$ is the set of matrices

$$\eta\uparrow(g)_{ij}^{i'j'} \equiv \delta(q_{i'}^{-1} g q_i, h \in H)\,\eta(h)_j^{j'}. \tag{6}$$

We now consider the set of vectors formed from the product of the group operator of the coset representatives O_q with the basis vectors of the irrep space of $\eta(H)$, $|\eta(H)j\rangle$, $j = 1, \ldots, |\eta|$. This set

$$\{O_{q_i}|\eta(H)j\rangle \equiv |\eta\uparrow ij\rangle \,|\, i = 1, \ldots, |G|/|H|, \; j = 1, \ldots, |\eta|\}, \tag{7}$$

with the orthonormality condition

$$\langle \eta\uparrow i'j' | \eta\uparrow ij\rangle = \delta_i^{i'} \delta_j^{j'}, \tag{8}$$

forms the basis of a $(|G|/|H|) \times |\eta|$-dimensional induced representation space. The group action on the basis vectors $|\eta\uparrow ij\rangle$ is determined *via* the group action $G: G/H \to G/H$, so that they transform according to the induced representation $\eta(H)\uparrow G$:

$$\begin{aligned} O_g|\eta\uparrow ij\rangle &= O_g O_{q_i}|\eta j\rangle \\ &= \sum_{i'} O_{q_{i'}}(O_{q_{i'}}^{-1} O_g O_{q_i})|\eta j\rangle \delta(q_{i'}^{-1} g q_i, h \in H) \\ &= \sum_{i'j'} |\eta\uparrow i'j'\rangle \eta\uparrow(g)_{ij}^{i'j'}. \end{aligned} \tag{9}$$

As a representation space of G, the space spanned by $\{|\eta\uparrow ij\rangle\}$ decomposes or subduces into irrep spaces of G labelled $\gamma(G)$,

$$|\eta\uparrow ij\rangle = \sum_{a\gamma l} |\eta\uparrow a\gamma l\rangle m(\eta\uparrow)_{ij}^{a\gamma l}. \tag{10}$$

Here a indexes the multiplicity of γ in $\eta\uparrow$, $|\eta\uparrow : \gamma|$, which by the Frobenius reciprocity theorem equals the multiplicity of η in γ, $|\gamma : \eta|$. The 'induction coefficient' $m(\eta\uparrow)_{ij}^{a\gamma l}$ is an element of an invertible matrix indexed by (ij) and $(a\gamma l)$

$$\begin{aligned} \sum_{ij} m(\eta\uparrow)_{ij}^{a'\gamma' l'} m(\eta\uparrow)^{-1}{}_{a\gamma l}^{ij} &= \delta_a^{a'} \delta_\gamma^{\gamma'} \delta_l^{l'} \\ \sum_{a\gamma l} m(\eta\uparrow)^{-1}{}_{a\gamma l}^{i'j'} m(\eta\uparrow)_{ij}^{a\gamma l} &= \delta_i^{i'} \delta_j^{j'}. \end{aligned} \tag{11}$$

We can choose the induction coefficients as (Haase & Dirl, 1987)

$$m(\eta\uparrow)_{ij}^{a\gamma l} = (|\gamma|/|G|)^{1/2} \gamma(q_i)_{a\eta j}^{l}. \tag{12}$$

The action of G on the new irrep basis vectors is given as

$$O_g|\eta\uparrow a\gamma l\rangle = \sum_{l'} |\eta\uparrow a\gamma l'\rangle \gamma(g)_l^{l'}. \tag{13}$$

We have choosen the irrep matrices of $\gamma(G)$ independently of the multiplicity label. We note that if the matrices $\eta(H)$ are unitary (or orthogonal) then $\eta(H)\uparrow G$ will also be unitary (or orthogonal). In this instance we have

$$m(\eta\uparrow)^{-1}{}_{a\gamma l}^{ij} = m(\eta\uparrow)_{ij}^{a\gamma l *}. \tag{14}$$

4. Projection operators on induced representation spaces

We now consider the action of the projection operators

$$P(\gamma)_l^{l'} = (|\gamma|/|G|) \sum_g \gamma(g^{-1})_l^{l'} O_g \tag{15}$$

on the basis vectors $|\eta\uparrow a\gamma l\rangle$ and $|\eta\uparrow ij\rangle$. The projection operators have the property

$$P(\gamma')_k^{k'} P(\gamma)_l^{l'} = \delta_\gamma^{\gamma'} \delta_l^{k'} P(\gamma)_k^{l'}. \tag{16}$$

From the great orthogonality theorem, we have

$$P(\gamma')_k^{k'}|\eta\uparrow a\gamma l\rangle = |\eta\uparrow a\gamma k\rangle \delta_\gamma^{\gamma'}\delta_l^{k'}; \tag{17}$$

or, equivalently, the projection operators can be formed from the irrep basis vectors,

$$P(\gamma)_l^{l'} = \sum_a |\eta\uparrow a\gamma l\rangle\langle\eta\uparrow a\gamma l'|. \tag{18}$$

On the basis vectors $|\eta\uparrow ij\rangle$ we have

$$P(\gamma)_l^{l'}|\eta\uparrow ij\rangle = \sum_a |\eta\uparrow a\gamma l\rangle m(\eta\uparrow)_{ij}^{a\gamma l'}. \tag{19}$$

On defining $P(\gamma)\equiv\sum_l P(\gamma)_l^l$ the above readily gives

$$P(\gamma')P(\gamma) = \delta_\gamma^{\gamma'}P(\gamma) \tag{20}$$

$$P(\gamma')|\eta\uparrow a\gamma l\rangle = |\eta\uparrow a\gamma l\rangle\delta_\gamma^{\gamma'} \tag{21}$$

$$P(\gamma) = \sum_{al} |\eta\uparrow a\gamma l\rangle\langle\eta\uparrow a\gamma l| \tag{22}$$

and

$$P(\gamma)|\eta\uparrow ij\rangle = \sum_{al} |\eta\uparrow a\gamma l\rangle m(\eta\uparrow)_{ij}^{a\gamma l}. \tag{23}$$

In what follows we shall use the projected basis vectors, $P(\gamma)|\eta\uparrow ij\rangle$, and we collect here some of their properties. Their scalar products determine the matrix elements of the projection operators $P(\gamma)$, which can also be given in terms of the induction coefficients

$$\begin{aligned}\langle\eta\uparrow i'j'|P(\gamma').P(\gamma)|\eta\uparrow ij\rangle &= \delta_\gamma^{\gamma'}\langle\eta\uparrow i'j'|P(\gamma)|\eta\uparrow ij\rangle\\ &= \sum_{al} m(\eta\uparrow)^{-1}{}_{a\gamma l}^{i'j'}\delta_\gamma^{\gamma'} m(\eta\uparrow)_{ij}^{a\gamma l}\\ &= (|\gamma|/|G|)\sum_a \gamma(q_{i'}^{-1}q_i)_{a\eta j}^{a\eta j'}\end{aligned} \tag{24}$$

or matrix elements of the products $q_{i'}^{-1}q_i$ by (12). Their scalar products with the irrep basis vectors determine the induction coefficients

$$\langle\eta\uparrow a\gamma l|\,.\,P(\gamma')|\eta\uparrow ij\rangle = \delta_{\gamma'}^{\gamma} m(\eta\uparrow)_{ij}^{a\gamma l}. \tag{25}$$

Furthermore, they transform under the group as $\eta(H)\uparrow G$:

$$O_g P(\gamma)|\eta\uparrow ij\rangle = \sum_{i'j'} P(\gamma)|\eta\uparrow i'j'\rangle\eta\uparrow(g)_{ij}^{i'j'}. \tag{26}$$

The number of projected basis vectors equals the dimension of the induced representations. It is in general larger than the dimension $|\gamma|$ of the irrep space. Therefore the projected basis vectors form a linearly dependent and non-orthogonal set of vectors. If both the representations $\eta(H)$ and $\gamma(G)$ are real orthogonal, all these objects are given in a real Euclidean space and allow for a geometric and crystallographic analysis. This analysis will be taken up in the following section.

5. Construction of polytopes in $\mathbb{E}^n$

Take a set of p vectors $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^p$ in $\mathbb{E}^n (p\geq n)$ and a selection of n of these vectors, say $\mathbf{e}^{a_1}, \mathbf{e}^{a_2}, \ldots, \mathbf{e}^{a_n}$, $a_i\in\{1,\ldots,p\}$. The volume spanned by these n vectors is given by the determinantal function

$$\det(\mathbf{e}^{a_1}\mathbf{e}^{a_2}\ldots\mathbf{e}^{a_n}) = \sum_{i_1 i_2\ldots i_n} \varepsilon_{i_1 i_2\ldots i_n} e_{i_1}^{a_1} e_{i_2}^{a_2}\ldots e_{i_n}^{a_n}, \tag{27}$$

where e_i^a is the ith component of $\mathbf{e}^a$ with respect to an orthogonal basis of $\mathbb{E}^n$ and $\varepsilon_{i_1 i_2\ldots i_n}$ is the generalized Levi-Civita symbol of $\mathbb{E}^n$. The properties of the determinant imply the following:

(i) $\det(\ldots\mathbf{e}^a\ldots\mathbf{e}^b\ldots) = -\det(\ldots\mathbf{e}^b\ldots\mathbf{e}^a\ldots)$ (28)

and hence if $\mathbf{e}^a=\mathbf{e}^b=\mathbf{e}$, $\det(\ldots\mathbf{e}\ldots\mathbf{e}\ldots)=0$;

(ii) if $\mathbf{Z}=\sum_a \mu_a\mathbf{e}^a$, $-\infty<\mu_a<+\infty$ then

$$\det(\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}\mathbf{Z}) = \sum_a \mu_a \det(\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}\mathbf{e}^a). \tag{29}$$

We note that the determinantal function takes positive, negative and zero values. If non-zero, the n-vector system defines a hypercell in $\mathbb{E}^n$; the sign of the determinant is associated with the relative orientation of the n-vector system which is said to be right(left)-handed if the sign is positive (negative). Equivalently, the sign determines whether the nth vector $\mathbf{e}^{a_n}$ is to the right or left of the hyperplane defined by the $(n-1)$-vector system $\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}$. If the determinant is zero, the n-vector system is said to be degenerate and forms a hyperplane in $\mathbb{E}^n$. Further classification of these hyperplanes can be made based on the rank of the determinant, *i.e.* the dimension of the largest non-zero subdeterminant. If the rank is $r<n$, then the hyperplane forms a hypercell in $\mathbb{E}^r$ embedded in $\mathbb{E}^n$.

Proposition 5.1: If $\mathbf{z}=\sum_a \mu_a\mathbf{e}^a$ with μ_a bounded, *i.e.* $\mu_a^{\min}\leq\mu_a\leq\mu_a^{\max}$ and $\{\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}\}$ span a hyperplane in $\mathbb{E}^n$ of rank $n-1$ then $\det(\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}\mathbf{z})$ is extremal when

$$\mu_a = \mu_a^{\text{ext}}\,\text{sign}\,[\det(\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}\mathbf{e}^a)]$$

where μ_a^{ext} takes one of the two possible values $\mu_a^{\max}, \mu_a^{\min}$. The vector

$$\mathbf{Z}^{\text{ext}} = \sum_a \mu_a^{\text{ext}}\,\text{sign}\,[\det(\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_{n-1}}\mathbf{e}^a)]\mathbf{e}^a$$

must clearly be constructed from vectors $\mathbf{e}^a$ which are not contained in the hyperplane. This is automatically ensured by the properties of the determinant. Furthermore, $\mathbf{Z}^{\text{ext}}$ is the largest vector not contained in the hyperplane.

Definition 5.1: A p-star in $\mathbb{E}^n$ with $p\geq n$ is a set of p unit vectors $(\mathbf{e}^1,\ldots,\mathbf{e}^p\}$ such that any subset of n vectors is linearly independent, *i.e.*

$$\det(\mathbf{e}^{a_1}\ldots\mathbf{e}^{a_n})\neq 0.$$

Definition 5.2: A p-line system associated with this p-star is the collection of p lines consisting of

$$\mathbf{x}=\tfrac{1}{2}\lambda_a\mathbf{e}^a \quad -1\leq\lambda_a\leq+1 \qquad a=1,\ldots,p.$$

We want to associate hypercells or polytopes to the p-line system and describe some characteristics of the polytopes in $\mathbb{E}^2$ and $\mathbb{E}^3$.

Definition 5.3: The polytope $\mathscr{H}(p)$ in $\mathbb{E}^n$ associated with the p-line system is the convex set of points

$$\mathscr{H}(p)=\{\mathbf{x}|\mathbf{x}=\tfrac{1}{2}\sum_{a=1}^{p}\lambda_a\mathbf{e}^a,\ -1\leq\lambda_b\leq+1,\ b=1,\ldots,p\}$$

centred at the origin.

Proposition 5.2: In $\mathbb{E}^2$ the polytope $\mathscr{H}(p)$ is a polygon with

$$\binom{p}{1}=p$$

pairs of parallel edges. For any vector $\mathbf{e}^s$ there are two parallel edges $\mathscr{E}_\pm(s)$ with

$$\mathscr{E}_\pm(s)=\{\mathbf{x}|\mathbf{x}=\pm\mathbf{y}(s)+\tfrac{1}{2}\lambda_s\mathbf{e}^s,\ -1\leq\lambda_s\leq1\}$$

where $\mathbf{y}(s)=\frac{1}{2}\sum_i^p \operatorname{sign}[\det(\mathbf{e}^s\mathbf{e}^i)]\mathbf{e}^i$.

Proof: Consider the line through the origin spanned by $\mathbf{e}^s$ and a vector from $\mathscr{H}(p)$ of the form

$$\mathbf{y}=\tfrac{1}{2}\sum_{a\neq s}\lambda_a\mathbf{e}^a.$$

The area determined by $\mathbf{y}$ and $\mathbf{e}^s$ is given by

$$\det(\mathbf{e}^s\mathbf{y})=\tfrac{1}{2}\sum\lambda_a\det(\mathbf{e}^a\mathbf{y}).$$

The extremal values of this area occur for the two choices

$$\lambda_a^{\text{ext}}=\pm\operatorname{sign}[\det(\mathbf{e}^a\mathbf{e}^s)]$$

and determine two points denoted $\pm\mathbf{y}(s)$, on the boundary of $\mathscr{H}(p)$. The points $\mathbf{x}=\frac{1}{2}\lambda_s\mathbf{e}^s\pm\mathbf{y}(s)$ are easily seen to form two parallel lines on the boundary of $\mathscr{H}(p)$. That is, they form the two parallel edges $\mathscr{E}_\pm(s)$.

Proposition 5.3: In $\mathbb{E}^3$ the polytope is a polyhedron with pairs of parallel polygon faces. For any plane defined by coplanar vectors $\{\mathbf{e}^{s_1},\ldots,\mathbf{e}^{s_r}\}$, there are two parallel faces of the polytope with points

$$\mathscr{F}_\pm(s_1\ldots s_r)=\{\mathbf{x}|\mathbf{x}=\pm\mathbf{y}(s_1\ldots s_r)+\tfrac{1}{2}\sum_{i=1}^{r}\lambda_{s_i}\mathbf{e}^{s_i},$$
$$-1\leq\lambda_s\leq+1\}$$

where $\mathbf{y}(s_1\ldots s_r)=\frac{1}{2}\sum_{a=1}^p\operatorname{sign}[\det(\mathbf{e}^{s_1}\mathbf{e}^{s_2}\mathbf{e}^a)]\mathbf{e}^a$. The numbers of different planes give the number of pairs of faces.

Proof: In $\mathbb{E}^3$ a plane of the polyhedron through the origin may be generated by two vectors $\mathbf{e}^{s_1}$ and $\mathbf{e}^{s_2}$. The other coplanar vectors are given by the set

$$\{\mathbf{e}^s|\det(\mathbf{e}^{s_1}\mathbf{e}^{s_2}\mathbf{e}^s)=0,\ s=1,\ldots,p\}.$$

Consider a point in $\mathscr{H}(p)$ but not in the plane

$$\mathbf{y}=\tfrac{1}{2}\sum_a\lambda_a\mathbf{e}^a \quad a\neq s_1,\ldots,s_r.$$

The extremal values of the vector $\mathbf{y}$ are given for the two choices

$$\lambda_a^{\text{ext}}=\pm\operatorname{sign}[\det(\mathbf{e}^{s_1}\mathbf{e}^{s_2}\mathbf{e}^a)]$$

where sign $[\det(\ldots)]$ determines whether $\mathbf{e}^a$ is to the right or left of the plane spanned by $\mathbf{e}^{s_1}$, $\mathbf{e}^{s_2}$. These two vectors denoted $\mathbf{y}(s_1\ldots s_r)$ fall on the boundary of $\mathscr{H}(p)$. The points

$$\mathbf{x}=\tfrac{1}{2}\sum_{i=1}^{r}\lambda_s\mathbf{e}^s\pm\mathbf{y}(s_1\ldots s_r)$$

also lie in the boundary of $\mathscr{H}(p)$ and from Proposition 5.2 form the two parallel polygon faces $\mathscr{F}_\pm(s_1\ldots s_r)$ of $\mathscr{H}(p)$.

Note that under the parity operation $\mathbf{e}^i\to-\mathbf{e}^i$ the edges $\mathscr{E}_\pm$ and faces $\mathscr{F}_\pm$ are mapped into $\mathscr{E}_\mp$ and $\mathscr{F}_\mp$ respectively. Thus all polygons and polyhedra contracted under Propositions 5.2 and 5.3 have central inversion symmetry [see Coxeter (1963), footnote to p. 28].

Part *B*: Application to the icosahedral group

We use the results of the previous sections in an application to the icosahedral group and its dihedral subgroup. We follow the procedure as outlined in Part *A*, giving:

(i) the orbit analysis of coset representatives $A(5)/D(m)$, $m=2,3,5$;

(ii) the representation theory of $A(5)$ and $D(m)$, in particular the induced representation $\tilde{0}[D(m)\uparrow A(5)]\equiv\tilde{0}\uparrow$ where $\tilde{0}[D(m)]$ is the one-dimensional non-trivial real orthogonal irrep of $D(m)$;

(iii) the projected basis vectors of $\tilde{0}\uparrow$ into the Euclidean space $\mathbb{E}^3$ determined by the three-dimensional irrep $[31^2_+]$ of $A(5)$;

(iv) the construction of polytopes from the star formed by these projected basis vectors, their description and characteristics.

6. Orbit analysis extending from the dihedral subgroups

The icosahedral group $A(5)$ consists of 60 elements with cyclic periods 2, 3 and 5. It can be generated from any two elements of different periods, *e.g.* g_2

and g_5 with conditions $g_5^5 = g_2^2 = g_3^3 = e$ and $g_3 = g_5^3 g_2 g_5^3$. The other elements are generated from products of g_2 and g_5 of the form

$$g_5^{\mu} g_2^{\sigma} \text{ and } g_5^{\mu} g_2 g_5^{\nu} g_2'^{\sigma} \quad \mu, \nu = 0, 1, \ldots, 4 \quad \sigma = 0, 1 \tag{30}$$

with $g_2' = g_2 g_5^2 g_2 g_5^3 g_2 g_5^2$ of period 2.

The elements g_5, g_3, g_2 generate respectively the cyclic subgroups $C(5)$, $C(3)$, $C(2)$. By including the element g_2', the larger dihedral group $D(m)$ of order $|D(m)| = 2m$, $m = 5, 3, 2$ are obtained:

$$D(m) = \{g_m^{\rho} g_2'^{\sigma} | \rho = 0, 1, \ldots, m-1 \quad \sigma = 0, 1\}. \tag{31}$$

As subgroups of $A(5)$, these dihedral groups will play an important role in what follows. A generating relation can be given for the set of coset representatives $A(5)/D(m)$ in terms of the generators g_2 and g_5. With $\mu = 0, 1, \ldots, 4$, we have

$$\begin{aligned} A(5)/D(5) &= \{e, g_5^{\mu} g_2\} \\ A(5)/D(3) &= \{g_5^{\mu}, g_5^{\mu} g_2 g_5^{-1}\} \\ A(5)/D(2) &= \{g_5^{\mu}, g_5^{\mu} g_2 g_5, g_5^{\mu} g_2 g_5^2\}. \end{aligned} \tag{32}$$

The number of coset representatives in each set is

$$|A(5)|/|D(m)| = 60/2m = 30/m = n.$$

With the explicit construction of $A(5)$, $D(m)$ and $A(5)/D(m)$ we are now able to perform the orbit analysis as given in § 2. To simplify enumeration we will use a single indexing of the coset representatives, instead of the above multiple indexing, obtained as follows

$$\begin{aligned} m=5 \quad & e = q_6, & g_5^{\mu-1} g_2 &= q_{\mu+1} \\ m=3 \quad & g_5^{\mu+2} = q_{\mu+1}, & g_5^{\mu} g_2 g_5^{-1} &= q_{\mu+6} \\ m=2 \quad & g_5^{\mu-1} = q_{\mu+1}, & g_5^{\mu} g_2 g_5 &= q_{\mu+6}, \\ & & g_5^{\mu} g_2 g_5^2 &= q_{\mu+11}. \end{aligned} \tag{33}$$

Under the group action defined in § 2, the stability group of X_1^n, *i.e.* of any one group element of period m, is then found to be $D(m)$. The orbit is then generated by application of the coset representatives of $A(5)/D(m)$, the total number being $n = 30/m$. For the other $X_p^n (1 < p \leq n)$, we make a similar analysis. A summary of results is given in Tables 1, 2 and 3, where for corresponding values of p orbit representatives of X_p^n are listed together with their stability groups and orbit characteristics. In compiling the tables complementarity of X_{n-p}^n and X_p^n is used. We note that the stability group for $X_n^n = A(5)/D(m)$ is $A(5)$ itself. The homomorphism permutes the q_i's and in this way an n-dimensional permutation representation can be given. If one employs the cycle notation for a permutation, the generators g_5, g_2 and g_3 then

Table 1. *Orbit representatives and stability groups for* X_p^6

$(p, n-p)$	Orbit representative	S_p^n	$\|A(5)\|/\|S_p^n\|$
0 6	() (123456)	$A(5)$	1
1 5	(1) (23456)	$D(5)$	6
2 4	(12) (3456)	$D(2)$	15
3 3	(123) (456)	$D(3)$	10

Table 2. *Orbit representatives and stability groups for* X_p^{10}

$(p, n-p)$	Orbit representative	S_p^n	$\|A(5)\|/\|S_p^n\|$
0 10	() (123456789 10)	$A(5)$	1
1 9	(1) (23456789 10)	$D(3)$	10
2 8	(12) (3456789 10)	$D(2)$	15
	(13) (2456789 10)	$C(2)$	30
3 7	(123) (456789 10)	$C(2)$	30
	(124) (356789 10)	$C(2)$	30
	(128) (345679 10)	$C(2)$	30
	(137) (245689 10)	$D(3)$	10
	(139) (245678 10)	$C(3)$	20
4 6	(1234) (56789 10)	$C(2)$	30
	(1236) (45789 10)	$C(2)$	30
	(1237) (45689 10)	$D(3)$	10
	(1239) (45678 10)	E	60
	(1248) (35679 10)	E	60
	(1249) (35678 10)	$D(2)$	15
	(139 10) (245678)	$A(4)$	5
5 5	(12345) (6789 10)	$D(5)$	6
	(12346) (5789 10)	E	60
	(1234 10) (56789)	$C(2)$	30
	(1239 10) (45678)	$C(2)$	30

take the form

	$n=6$	$n=10$
g_5	(1 2 3 4 5)(6)	(1 2 3 4 5)(6 7 8 9 10)
g_2	(13)(26)(4)(5)	(12)(36)(49)(57)(8)(10)
g_3	(123)(465)	(178)(2 10 6)(359)(4)

	$n=15$
g_5	(1 2 3 4 5)(6 7 8 9 10)(11 12 13 14 15)
g_2	(17)(2)(36)(4 14)(5 12)(8 11)(9)(10 15)(13)
g_3	(1 12 8)(2 15 11)(3 10 14)(459)(6 7 13).

7. Representation theory of A(5)

The representation theory of $A(5)$ and its subgroups $D(m)$, $m = 5, 3, 2$, is well-known. In Tables 4-7 we give the irrep labels and their characters. The columns of Table 8 give the subduction rules for the $A(5)$ irrep under the branching $A(5) \supset D(m)$. By the Frobenius reciprocity theorem the irrep content of representations of $A(5)$ induced from irreps of $D(m)$ are simply read off as the rows of Tables 4-8. Note no branching multiplicity occurs.

Since we want to consider results in $\mathbb{E}^3$ we focus attention on the appearance of the three-dimensional irrep $[31_+^2]$ of $A(5)$. Moreover, for reasons of simplicity we select only one-dimensional irreps of the dihedral groups. Both restrictions together single out three specific representations induced from $D(m)$, *i.e.* $\tilde{0}[D(m)\uparrow A(5)] \equiv \tilde{0}\uparrow$. The dimension of $\tilde{0}\uparrow$ is $n = |A(5)|/|D(m)|$ and has matrix elements

$$\tilde{0}\uparrow(g)_{i0}^{i'0} = \tilde{0}(h) \times \delta(q_{i'}^{-1} g q_i, h \in D(m)). \tag{34}$$

Table 3. *Orbit representatives and stability groups for* X_p^{15}

$(p, n-p)$	S_p^n		Orbit representatives – (c) denotes the complementary subset
0 15	$A(5)$	1	(–)(*c*)
1 14	$D(2)$	15	(1)(*c*)
2 13	$D(2)$	15	(1 8)(*c*)
	$C(2)$	30	(1 2)(*c*), (1 3)(*c*), (1 7)(*c*)
3 12	$A(4)$	5	(1 8 12)(*c*)
	$D(3)$	10	(1 2 6)(*c*), (1 3 13)(*c*), (1 7 9)(*c*)
	$C(2)$	30	(1 2 3)(*c*), (1 2 4)(*c*), (1 2 7)(*c*), (1 2 15)(*c*), (1 3 9)(*c*), (1 3 11)(*c*)
	E	60	(1 2 8)(*c*), (1 2 11)(*c*), (1 2 12)(*c*), (1 3 12)(*c*)
4 11	$D(2)$	15	(1 2 11 14)(*c*)
	$C(2)$	30	(1 2 3 4)(*c*), (1 2 3 9)(*c*), (1 2 3 13)(*c*), (1 2 4 6)(*c*), (1 2 4 15)(*c*), (1 2 6 8)(*c*), (1 2 7 9)(*c*), (1 2 7 10)(*c*), (1 2 7 13)(*c*), (1 2 8 9)(*c*), (1 2 8 13)(*c*), (1 2 12 13)(*c*), (1 3 9 11)(*c*), (1 3 9 13)(*c*), (1 3 11 12)(*c*)
	E	60	(1 2 3 6)(*c*), (1 2 3 8)(*c*), (1 2 3 11)(*c*), (1 2 3 12)(*c*), (1 2 4 7)(*c*), (1 2 4 8)(*c*), (1 2 4 11)(*c*), (1 2 4 12)(*c*), (1 2 7 8)(*c*), (1 2 8 11)(*c*), (1 2 8 12)(*c*), (1 2 8 14)(*c*), (1 2 8 15)(*c*), (1 2 11 12)(*c*), (1 2 11 15)(*c*)
5 10	$D(5)$	6	(1 2 3 4 5)(*c*), (1 2 7 10 15)(*c*), (1 3 9 11 15)(*c*)
	$D(2)$	15	(1 2 3 6 7)(*c*), (1 2 4 11 14)(*c*), (1 2 11 14 15)(*c*)
	$C(2)$	30	(1 2 3 4 7)(*c*), (1 2 3 4 11)(*c*), (1 2 3 6 11)(*c*), (1 2 3 8 10)(*c*), (1 2 3 8 13)(*c*), (1 2 3 9 13)(*c*), (1 2 3 11 15)(*c*), (1 2 3 12 14)(*c*), (1 2 4 6 12)(*c*), (1 2 4 6 15)(*c*), (1 2 4 7 10)(*c*), (1 2 4 7 14)(*c*), (1 2 4 12 13)(*c*), (1 2 6 8 9)(*c*), (1 2 7 8 11)(*c*), (1 2 7 9 13)(*c*), (1 2 8 9 15)(*c*), (1 2 8 13 14)(*c*)
	E	60	(1 2 3 4 6)(*c*), (1 2 3 4 9)(*c*), (1 2 3 4 12)(*c*), (1 2 3 4 13)(*c*), (1 2 3 6 8(*c*), (1 2 3 6 9)(*c*), (1 2 3 6 10)(*c*), (1 2 3 6 12)(*c*), (1 2 3 6 13)(*c*), (1 2 3 6 15)(*c*), (1 2 3 8 9)(*c*), (1 2 3 8 11)(*c*), (1 2 3 8 12)(*c*), (1 2 3 8 14)(*c*), (1 2 3 8 15)(*c*), (1 2 3 9 11)(*c*), (1 2 3 9 12)(*c*), (1 2 3 11 12)(*c*), (1 2 3 11 13)(*c*), (1 2 3 11 14)(*c*), (1 2 3 12 13)(*c*), (1 2 4 6 8)(*c*), (1 2 4 7 8)(*c*), (1 2 4 7 9)(*c*), (1 2 4 7 11)(*c*), (1 2 4 7 13)(*c*), (1 2 4 7 15)(*c*), (1 2 4 8 13)(*c*), (1 2 4 8 13)(*c*), (1 2 4 8 14)(*c*), (1 2 4 8 15)(*c*), (1 2 4 11 12)(*c*), (1 2 4 11 15)(*c*), (1 2 4 12 15)(*c*), (1 2 7 8 9)(*c*), (1 2 7 8 10)(*c*), (1 2 8 9 12)(*c*), (1 2 8 11 12)(*c*), (1 2 8 11 15)(*c*), (1 2 8 14 15)(*c*)
6 9	$D(3)$	10	(1 2 3 8 10 13)(*c*), (1 2 4 6 12 13)(*c*), (1 2 6 8 9 15)(*c*), (1 2 8 9 12 13)(*c*)
	$D(2)$	15	(1 2 3 6 7 9)(*c*), (1 2 3 6 7 13)(*c*), (1 2 4 11 14 15)(*c*)
	$C(3)$	20	(1 2 3 6 10 14)(*c*), (1 2 4 8 12 14)(*c*), (1 2 8 13 14 15)(*c*)
	$C(2)$	30	(1 2 3 4 5 6)(*c*), (1 2 3 4 5 11)(*c*), (1 2 3 4 6 8)(*c*), (1 2 3 4 7 11)(*c*), (1 2 3 4 9 10)(*c*), (1 2 3 4 12 15)(*c*), (1 2 3 4 13 14)(*c*), (1 2 3 6 8 11)(*c*), (1 2 3 6 11 12)(*c*), (1 2 3 6 8 10)(*c*), (1 2 3 8 11 14)(*c*), (1 2 3 8 13 14)(*c*), (1 2 3 9 11 14)(*c*), (1 2 3 9 11 15)(*c*), (1 2 3 9 12 14)(*c*), (1 2 3 11 13 15)(*c*), (1 2 3 12 13 14)(*c*), (1 2 4 6 8 13)(*c*), (1 2 4 7 8 13)(*c*), (1 2 4 7 8 15)(*c*), (1 2 4 7 10 15)(*c*), (1 2 7 8 9 10)(*c*), (1 2 7 8 9 11)(*c*)
	E	60	(1 2 3 4 6 7)(*c*), (1 2 3 4 6 9)(*c*), (1 2 3 4 6 10)(*c*), (1 2 3 4 6 11)(*c*), (1 2 3 4 6 12)(*c*), (1 2 3 4 6 13)(*c*), (1 2 3 4 6 14)(*c*), (1 2 3 4 6 15)(*c*), (1 2 3 4 7 9)(*c*), (1 2 3 4 7 12)(*c*), (1 2 3 4 7 13)(*c*), (1 2 3 4 9 11)(*c*), (1 2 3 4 9 12)(*c*), (1 2 3 4 9 13)(*c*), (1 2 3 4 9 14)(*c*), (1 2 3 4 9 15)(*c*), (1 2 3 4 11 12)(*c*), (1 2 3 4 11 13)(*c*), (1 2 3 4 12 13)(*c*), (1 2 3 4 12 14)(*c*), (1 2 3 6 7 8)(*c*), (1 2 3 6 8 9)(*c*), (1 2 3 6 8 10)(*c*), (1 2 3 6 8 12)(*c*), (1 2 3 6 8 13)(*c*), (1 2 3 6 8 14)(*c*), (1 2 3 6 8 15)(*c*), (1 2 3 6 9 10)(*c*), (1 2 3 6 9 11)(*c*), (1 2 3 6 9 12)(*c*), (1 2 3 6 9 13)(*c*), (1 2 3 6 9 15)(*c*), (1 2 3 6 10 12)(*c*), (1 2 3 6 12 13)(*c*), (1 2 3 6 12 15)(*c*), (1 2 3 6 13 15)(*c*), (1 2 3 8 9 11)(*c*), (1 2 3 8 9 12)(*c*), (1 2 3 8 9 13)(*c*), (1 2 3 8 9 14)(*c*), (1 2 3 8 9 15)(*c*), (1 2 3 8 10 11)(*c*), (1 2 3 8 10 12)(*c*), (1 2 3 8 11 12)(*c*), (1 2 3 8 11 15)(*c*), (1 2 3 8 12 14)(*c*), (1 2 3 8 12 15)(*c*), (1 2 3 8 14 15)(*c*), (1 2 3 9 11 12)(*c*), (1 2 3 9 11 13)(*c*), (1 2 3 9 12 13)(*c*), (1 2 3 11 12 13)(*c*), (1 2 3 11 12 14)(*c*), (1 2 3 11 12 15)(*c*), (1 2 3 11 13 14)(*c*), (1 2 4 6 8 12)(*c*), (1 2 4 6 8 15)(*c*), (1 2 4 7 8 10)(*c*), (1 2 4 7 8 14)(*c*), (1 2 4 7 9 11)(*c*), (1 2 4 7 9 13)(*c*), (1 2 4 7 9 15)(*c*), (1 2 4 7 10 11)(*c*), (1 2 4 7 11 13)(*c*), (1 2 4 7 11 15)(*c*), (1 2 4 8 12 13)(*c*), (1 2 4 8 12 15)(*c*), (1 2 4 8 14 15)(*c*), (1 2 4 11 12 15)(*c*)

Table 3 (*cont.*)

$(p, n-p)$	S_p^n		Orbit representatives – (c) denotes the complementary subset
7 8	$D(2)$	15	(1 2 3 6 7 9 13)(c)
	$C(2)$	30	(1 2 3 4 5 6 7)(c), (1 2 3 4 5 6 8)(c), (1 2 3 4 5 6 15)(c),
			(1 2 3 4 5 11 12)(c), (1 2 3 4 5 11 13)(c), (1 2 3 4 6 7 8)(c),
			(1 2 3 4 6 7 12)(c), (1 2 3 4 6 7 14)(c), (1 2 3 4 6 8 11)(c),
			(1 2 3 4 6 11 13)(c), (1 2 3 4 7 9 10)(c), (1 2 3 4 7 12 15)(c),
			(1 2 3 4 7 13 14)(c), (1 2 3 4 9 10 11)(c), (1 2 3 4 9 11 14)(c),
			(1 2 3 4 11 12 15)(c), (1 2 3 4 11 13 14)(c), (1 2 3 6 7 8 10)(c),
			(1 2 3 6 7 8 11)(c), (1 2 3 6 8 11 12)(c), (1 2 3 6 9 11 15)(c),
			(1 2 3 8 9 10 13)(c), (1 2 3 8 9 12 13)(c), (1 2 3 8 10 11 15)(c),
			(1 2 3 8 10 12 14)(c), (1 2 3 8 11 14 15)(c), (1 2 3 9 11 12 15)(c),
			(1 2 3 9 11 13 15)(c), (1 2 3 9 12 13 14)(c), (1 2 3 11 12 14 15)(c),
			(1 2 4 6 8 12 13)(c), (1 2 4 6 8 12 15)(c), (1 2 4 7 8 10 15)(c),
			(1 2 4 7 8 13 14)(c), (1 2 4 7 8 14 15)(c), (1 2 4 7 10 11 15)(c)
	E	60	(1 2 3 4 5 6 11)(c), (1 2 3 4 5 6 12)(c), (1 2 3 4 6 7 9)(c),
			(1 2 3 4 6 7 10)(c), (1 2 3 4 6 7 11)(c), (1 2 3 4 6 7 13)(c),
			(1 2 3 4 6 7 15)(c), (1 2 3 4 6 8 9)(c), (1 2 3 4 6 8 12)(c),
			(1 2 3 4 6 8 13)(c), (1 2 3 4 6 9 10)(c), (1 2 3 4 6 9 11)(c),
			(1 2 3 4 6 9 12)(c), (1 2 3 4 6 9 13)(c), (1 2 3 4 6 9 14)(c),
			(1 2 3 4 6 9 15)(c), (1 2 3 4 6 10 11)(c), (1 2 3 4 6 10 12)(c),
			(1 2 3 4 6 10 13)(c), (1 2 3 4 6 10 14)(c), (1 2 3 4 6 10 15)(c),
			(1 2 3 4 6 11 12)(c), (1 2 3 4 6 11 14)(c), (1 2 3 4 6 11 15)(c),
			(1 2 3 4 6 12 13)(c), (1 2 3 4 6 12 15)(c), (1 2 3 4 6 13 14)(c),
			(1 2 3 4 6 13 15)(c), (1 2 3 4 6 14 15)(c), (1 2 3 4 7 9 11)(c),
			(1 2 3 4 7 9 12)(c), (1 2 3 4 7 9 13)(c), (1 2 3 4 7 9 14)(c),
			(1 2 3 4 7 9 15)(c), (1 2 3 4 7 11 12)(c), (1 2 3 4 7 11 13)(c),
			(1 2 3 4 7 12 13)(c), (1 2 3 4 7 12 14)(c), (1 2 3 4 9 10 12)(c),
			(1 2 3 4 9 10 13)(c), (1 2 3 4 9 11 12)(c), (1 2 3 4 9 11 13)(c),
			(1 2 3 4 9 11 15)(c), (1 2 3 4 9 12 13)(c), (1 2 3 4 9 12 14)(c),
			(1 2 3 4 9 12 15)(c), (1 2 3 4 9 13 14)(c), (1 2 3 4 9 13 15)(c),
			(1 2 3 4 9 14 15)(c), (1 2 3 4 11 12 13)(c), (1 2 3 4 11 12 14)(c),
			(1 2 3 4 12 13 14)(c), (1 2 3 4 12 13 15)(c), (1 2 3 6 7 8 9)(c),
			(1 2 3 6 7 9 11)(c), (1 2 3 6 8 9 10)(c), (1 2 3 6 8 9 11)(c),
			(1 2 3 6 8 9 12)(c), (1 2 3 6 8 9 13)(c), (1 2 3 6 8 9 14)(c),
			(1 2 3 6 8 9 15)(c), (1 2 3 6 8 10 12)(c), (1 2 3 6 8 10 14)(c),
			(1 2 3 6 8 12 13)(c), (1 2 3 6 8 13 14)(c), (1 2 3 6 8 14 15)(c),
			(1 2 3 6 9 10 12)(c), (1 2 3 6 9 11 12)(c), (1 2 3 6 9 12 13)(c),
			(1 2 3 6 9 12 15)(c), (1 2 3 6 9 13 15)(c), (1 2 3 6 12 13 15)(c),
			(1 2 3 8 9 10 11)(c), (1 2 3 8 9 10 12)(c), (1 2 3 8 9 11 12)(c),
			(1 2 3 8 9 11 15)(c), (1 2 3 8 9 12 14)(c), (1 2 3 8 9 12 15)(c),
			(1 2 3 8 9 13 14)(c), (1 2 3 8 9 14 15)(c), (1 2 3 8 10 11 12)(c),
			(1 2 3 8 10 11 14)(c), (1 2 3 8 11 12 15)(c), (1 2 3 8 12 14 15)(c),
			(1 2 3 8 11 12 13)(c), (1 2 3 8 11 12 14)(c), (1 2 3 11 12 13 14)(c),
			(1 2 3 11 13 14 15)(c), (1 2 4 7 9 11 13)(c), (1 2 4 7 9 11 15)(c),
			(1 2 4 8 12 13 14)(c)

The value of $\tilde{0}(h)=\pm 1$ is obtained from Table 4. Since $\tilde{0}$ is a real orthogonal irrep, the induced representation $\tilde{0}\uparrow$ is also real orthogonal. The representation space can then be embedded into $\mathbb{E}^n$. With a simplification of notation we have from § 3 that the basis vectors

$$\{|\tilde{0}\uparrow i0\rangle \equiv {}^m\mathbf{c}_i \quad i=1,\ldots,n\} \tag{35}$$

provide an orthogonal basis, ${}^m\mathbf{c}_i \,.\, {}^m\mathbf{c}_j=\delta_{ij}$, for the representation $\tilde{0}\uparrow$. Furthermore the irrep basis vectors

$$\{|\tilde{0}\uparrow 0\gamma l\rangle \equiv {}^m\mathbf{d}_{\gamma l} | \gamma \text{ an irrep of } A(5),\ l=1,\ldots,|\gamma|\}, \tag{36}$$

where γ is given by Table 8, are related by the orthogonal matrix, $m(\tilde{0}\uparrow)^{0\gamma l}_{i0}\equiv m^{\gamma l}_i$, $m^{-1\,i}{}_{\gamma l}=m^{\gamma l}_i$,

$$ {}^m\mathbf{d}_{\gamma l}=\sum_i {}^m\mathbf{c}_i m^{\gamma l}_i \tag{37}$$

$$ {}^m\mathbf{c}_i=\sum_{\gamma l} {}^m\mathbf{d}_{\gamma l} m^{\gamma l}_i . \tag{38}$$

The projection of the basis vectors ${}^m\mathbf{c}_i$ onto the three-dimensional irrep space labelled by $\gamma=[31^2_+]$ is written

$$P(31^2_+)\,.\,{}^m\mathbf{c}_i \equiv \sqrt{\mathcal{N}(m)}\,{}^m\mathbf{e}_i=\sum_l {}^m\mathbf{d}_{[31^2_+]l} m_i^{[31^2_+]l}, \tag{39}$$

where $\mathcal{N}(m)=m/10$ is a scaling factor normalizing the vectors ${}^m\mathbf{e}_i$ to unit length, ${}^m\mathbf{e}_i \,.\, {}^m\mathbf{e}_i=+1$. The matrix elements of the projection operator $P(31^2_+)$ are given by (24):

$$\langle \tilde{0}\uparrow i'0|P(31^2_+)|\tilde{0}\uparrow i0\rangle=\mathcal{N}(m)\,{}^m\mathbf{e}_{i'} \,.\, {}^m\mathbf{e}_i, \tag{40}$$

and if the vectors ${}^m\mathbf{d}_{\gamma l}$ are given then the induction coefficients are, by (25),

$$m^{\gamma l}_i=\sqrt{\mathcal{N}(m)}\,{}^m\mathbf{d}_{\gamma l} \,.\, {}^m\mathbf{e}_i . \tag{41}$$

We can now link the n-star and n-line system of § 5 with the projected basis vectors of the induced representation space of $\tilde{0}\uparrow$ and the orbit analysis of $A(5)/D(m)$.

Proposition 7.1: In the representation space $\mathbb{E}^n$ of the induced representation $\tilde{0}\uparrow$, the irrep $\gamma=[31^2_+]$ deter-

Table 4. *Irreducible representations and characters of* $A(5)$

Class	Rep.	No.	Irreducible representations [5]	[41]	[32]	$[31^2_+]$	$[31^2_-]$
1	e	1	1	4	5	3	3
2	g_5	12	1	−1	0	φ	$1-\varphi$
3	g_5^2	12	1	−1	0	$1-\varphi$	φ
4	g_3	20	1	+1	−1	0	0
5	g_2'	15	1	0	1	−1	−1

$\varphi=\frac{1}{2}(\sqrt{5}+1)$ $1-\varphi=-\frac{1}{2}(\sqrt{5}-1)$

Table 5. *Irreducible representations and characters of* $D(5)$

Class	Rep.	No.	Irreducible representations 0	$\tilde{0}$	1	2
1	e	1	1	1	2	2
2	g_5	2	1	1	$\varphi-1$	$-\varphi$
3	$(g_5)^2$	2	1	1	$-\varphi$	$\varphi-1$
4	g_2'	5	1	−1	0	0

Table 6. *Irreducible representations and characters of* $D(3)$

Class	Rep.	No.	Irreducible representation 0	$\tilde{0}$	1
1	e	1	1	1	2
2	g_3	2	1	1	−1
3	g_2'	3	1	−1	0

Table 7. *Irreducible representations and characters of* $D(2)$

Class	Rep.	No.	Irreducible representation 0	$\tilde{0}$	1	$\tilde{1}$
1	e	1	1	1	1	1
2	g_2	1	1	1	−1	−1
3	g_2'	1	1	−1	1	−1
4	g_2g_2'	1	1	−1	−1	1

Table 8. *Subduction of irreducible representations from* $A(5)$ *to* $D(m)$

		$A(5)$: [5]	[41]	[32]	$[31^2_+]$	$[31^2_-]$
$D(5)$:	0	1	0	1	0	0
	$\tilde{0}$	0	0	0	1	1
	1	0	1	1	1	0
	2	0	1	1	0	1
$D(3)$:	0	1	1	1	0	0
	$\tilde{0}$	0	1	0	1	1
	1	0	1	2	1	1
$D(2)$:	0	1	1	2	0	0
	$\tilde{0}$	0	1	1	1	1
	1	0	1	1	1	1
	$\tilde{1}$	0	1	1	1	1

mines an orthogonal subspace $\mathbb{E}^3$ and its orthogonal complement $\mathbb{E}^{n-3}$. The n projected unit vectors ${}^m\mathbf{e}_i$ in $\mathbb{E}^3$ form an n-star as defined in Definition 5.1 and determine an n-line system as given by Definition 5.2. Proposition 5.3 can be employed to construct various polyhedra from the n-line system. In addition the vectors ${}^m\mathbf{e}_i$ are themselves enumerated by the coset representatives q_i of $A(5)/D(m)$ and provide a representation of $A(5)$ under the group action. As a consequence the various p-stars, p-line systems and their polyhedra for $1\le p\le n$ may be classified by the orbit analysis of $A(5)/D(m)$ given in Table 1.

8. Families of icosahedral polyhedra

We now give an explicit geometric form for the n-stars ${}^m\mathbf{e}_i$ in $\mathbb{E}^3$. We can then employ the methods of § 5 to construct or produce figures of the various polyhedra.

The stability group of the p-line determined by ${}^m\mathbf{e}_i$ is $D(m)$. In $\mathbb{E}^3$, the three n-stars $\{{}^m\mathbf{e}_i\}$, $m=5,3,2$ determine three systems of n-lines and form the m-fold axes of rotation of a dodecahedron. Fig. 1 shows the enumeration of the axes according to the n-star. This enumeration of the fivefold axes coincides with that of Kramer & Neri (1984). In Table 9 we specify the vectors along the fivefold axes choosing ${}^5\mathbf{e}_6$ along the 3-axis and ${}^5\mathbf{e}_1$ lying in the 1–3 plane. The vectors along the threefold axes ${}^3\mathbf{e}_i$ and twofold axes ${}^2\mathbf{e}_i$ may be expressed as linear combinations of ${}^5\mathbf{e}_i$. We have

$$ {}^3\mathbf{e}_i=\mathcal{N}_{35}(\pm{}^5\mathbf{e}_{i_1}\pm{}^5\mathbf{e}_{i_2}\pm{}^5\mathbf{e}_{i_3}) \quad \mathcal{N}_{35}=[\tfrac{5}{9}\varphi^{-6}]^{1/4} \tag{42} $$

$$ {}^2\mathbf{e}_i=\mathcal{N}_{25}(\pm{}^5\mathbf{e}_{i_1}\pm{}^5\mathbf{e}_{i_2}) \quad \mathcal{N}_{25}=[\tfrac{5}{16}\varphi^{-2}]^{1/4} \tag{43} $$

$$ \text{and} \quad \varphi=\tfrac{1}{2}(\sqrt{5}+1). $$

The correspondence between the index of threefold axes (i) and those of the fivefold axes ($i_1i_2i_3$) is given in Table 10. Similarly, the correspondence between two- and fivefold axes is given in Table 11. The notation $\bar{i}_a$ used in these tables implies a negative sign for the vector, *i.e.* the appearance of $-{}^5\mathbf{e}_i$.

The vectorial representation makes it easy to obtain the angular relation between the vectors ${}^m\mathbf{e}_i$ and

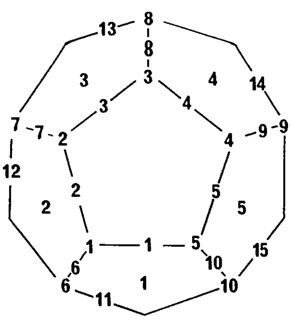

Fig. 1. The enumeration of the fivefold, threefold and twofold axes of rotation of a dodecahedron oriented face first.

Table 9. *The* ${}^5\mathbf{e}_i$ *vectors*

${}^5\mathbf{e}_1$	${}^5\mathbf{e}_2$	${}^5\mathbf{e}_3$	${}^5\mathbf{e}_4$	${}^5\mathbf{e}_5$	${}^5\mathbf{e}_6$
2γ	$2\gamma\alpha$	$2\gamma\alpha'$	$2\gamma\alpha'$	$2\gamma\alpha$	0
0	$-2\gamma\beta$	$-2\gamma\beta'$	$2\gamma\beta'$	$2\gamma\beta$	0
γ	γ	γ	γ	γ	1

$\gamma=\sqrt{(1/5)}$ $\qquad \alpha=\cos\theta=\frac{1}{4}(\sqrt{5}-1)$ $\qquad \beta=\sin\theta$

$\theta=2\pi/5$ $\qquad \alpha'=\cos 2\theta=-\frac{1}{4}(\sqrt{5}+1)$ $\qquad \beta'=\sin 2\theta$

Table 10. *The* ${}^3\mathbf{e}_i=\mathcal{N}_{35}(\pm{}^5\mathbf{e}_{i_1}\pm{}^5\mathbf{e}_{i_2}\pm{}^5\mathbf{e}_{i_3})$ *correspondence*

i	1	2	3	4	5	6	7	8	9	10
$i_1i_2i_3$	126	236	346	456	516	$12\bar{4}$	$23\bar{5}$	$34\bar{1}$	$45\bar{2}$	$51\bar{3}$

Table 11. *The* ${}^3\mathbf{e}_i=\mathcal{N}_{25}(\pm{}^5\mathbf{e}_{i_1}\pm{}^5\mathbf{e}_{i_2})$ *correspondence*

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
i_1i_2	16	26	36	46	56	12	23	34	45	51	$1\bar{4}$	$2\bar{5}$	$3\bar{1}$	$4\bar{2}$	$5\bar{3}$

Table 12. *Matrix form of* ${}^5\mathbf{e}_i\,.\,{}^5\mathbf{e}_j$

${}^5\mathbf{e}_i\,.\,{}^5\mathbf{e}_j$	1	2	3	4	5	6
1	1	a	$-a$	$-a$	a	a
2		1	a	$-a$	$-a$	a
3			1	a	$-a$	a
4				1	a	a
5					1	a
6						1

$a=\sqrt{(1/5)}$ $\qquad \mathcal{N}(5)=\frac{1}{2}$

Table 13. *Matrix form of* ${}^3\mathbf{e}_i\,.\,{}^3\mathbf{e}_j$

${}^3\mathbf{e}_i\,.\,{}^3\mathbf{e}_j$	1	2	3	4	5	6	7	8	9	10
1	1	b_1	b_2	b_2	b_1	b_1	b_2	$-b_2$	$-b_2$	b_2
2		1	b_1	b_2	b_2	b_2	b_1	b_2	$-b_2$	$-b_2$
3			1	b_1	b_2	$-b_2$	b_2	b_1	b_2	$-b_2$
4				1	b_1	$-b_2$	$-b_2$	b_2	b_1	b_2
5					1	b_2	$-b_2$	$-b_2$	b_2	b_1
6						1	b_2	$-b_1$	$-b_1$	b_2
7							1	b_2	$-b_1$	$-b_1$
8								1	b_2	$-b_1$
9									1	b_2
10										1

$b_1=(\sqrt{5})/3$ $\quad b_2=1/3$ $\quad \mathcal{N}(3)=3/10$

Table 14. *Matrix form of* ${}^2\mathbf{e}_i\,.\,{}^2\mathbf{e}_j$

${}^2\mathbf{e}_i\,.\,{}^2\mathbf{e}_j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	c_1	c_2	c_2	c_1	c_1	c_3	c_4	c_3	c_1	c_2	c_4	$-c_2$	c_3	c_3
2		1	c_1	c_2	c_2	c_1	c_1	c_3	c_4	c_3	c_3	c_2	c_4	$-c_2$	c_3
3			1	c_1	c_2	c_3	c_1	c_1	c_3	c_4	c_3	c_3	c_2	c_4	$-c_2$
4				1	c_1	c_4	c_3	c_1	c_1	c_3	$-c_2$	c_3	c_3	c_2	c_4
5					1	c_3	c_4	c_3	c_1	c_1	c_4	$-c_2$	c_3	c_3	c_2
6						1	c_2	$-c_3$	$-c_3$	c_2	c_1	c_2	$-c_2$	$-c_1$	c_4
7							1	c_2	$-c_3$	$-c_3$	c_4	c_1	c_2	$-c_2$	$-c_1$
8								1	c_2	$-c_3$	$-c_1$	c_4	c_1	c_2	$-c_2$
9									1	c_2	$-c_2$	$-c_1$	c_4	c_1	c_2
10										1	c_2	$-c_2$	$-c_1$	c_4	c_1
11											1	c_3	$-c_1$	$-c_1$	c_3
12												1	c_3	$-c_1$	$-c_1$
13													1	c_3	$-c_1$
14														1	c_3
15															1

$c_1=\frac{1}{4}(\sqrt{5}+1)$ $\quad c_2=\frac{1}{2}$ $\quad c_3=\frac{1}{4}(\sqrt{5}-1)$ $\quad c_4=0$

$\mathcal{N}(2)=\frac{1}{5}$

${}^m\mathbf{e}_j(i\neq j)$ from $\cos({}^m\theta_{ij})={}^m\mathbf{e}_i\,.\,{}^m\mathbf{e}_j$. These scalar products are given in matrix form in Tables 12–14. Note that from the orbit analysis of $A(5)/D(m)$, we have as many different angles ${}^m\theta_{ij}$ as there are orbits of X_2^n. Choosing for $m=5,3,2$ the orbit representatives of the X_2^n from Tables 1-3, one finds

$$\cos({}^5\theta_{12})=\sqrt{1/5} \qquad {}^5\theta_{12}\simeq 63{\cdot}4^\circ$$
$$\cos({}^3\theta_{12})=\sqrt{5/9} \qquad {}^3\theta_{12}\simeq 41{\cdot}81^\circ$$
$$\cos({}^3\theta_{12})=1/3 \qquad {}^3\theta_{13}\simeq 70{\cdot}52^\circ$$
$$\cos({}^2\theta_{12})=\tfrac{1}{2}\varphi \qquad {}^2\theta_{12}=36^\circ$$
$$\cos({}^2\theta_{13})=\tfrac{1}{2} \qquad {}^2\theta_{13}=60^\circ$$
$$\cos({}^2\theta_{17})=\tfrac{1}{2}\varphi^{-1} \qquad {}^2\theta_{17}=72^\circ$$
$$\cos({}^2\theta_{18})=0 \qquad {}^2\theta_{18}=90^\circ.$$

Tables 12–14 determine according to (40) the $n\times n$ matrix of the projection operator $P(31_+^2)$ provided that they are multiplied by the scalar factor $\mathcal{N}(m)$ which takes the length of the projected vectors into account. The scalar factors are given with the tables.

If we give an explicit form for the irrep basis vectors labelled by $\gamma=[31_+^2]$ as

$$ {}^m\mathbf{d}_{[31_+^2]1}=\begin{pmatrix}1\\0\\0\end{pmatrix} \quad {}^m\mathbf{d}_{[31_+^2]2}=\begin{pmatrix}0\\1\\0\end{pmatrix} \quad {}^m\mathbf{d}_{[31_+^2]3}=\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{44}$$

then the induction coefficients $m_i^{[31_+^2]l}$ are by (41) just the entries of the $3\times n$ matrix given by Table 9 multiplied by $\mathcal{N}(m)^{1/2}$. Other orthogonal bases exist for the irrep space of $[31_+^2]$; however, the above is simple and convenient.

In constructing the polyhedra $\mathcal{H}(p)$ given by Proposition 5.3, we use the fact that the symmetry group of $\mathcal{H}(p)$ is the same as the stability group S_p^n of the corresponding p-line system. It therefore suffices to construct the polyhedra corresponding to orbit representatives of Table 1. All other polyhedra are obtained by the application of icosahedral rotations from $A(5)/S_p^n$. Note that although the polyhedra have central inversion symmetry (see § 5) this is not a symmetry-group element of $A(5)$.

In general the polyhedra are composite structures constructed from elementary polyhedra. In the cases $m=5,3$, we find no coplanar sets of three or more vectors, and hence all faces of the polyhedra are rhombic. Furthermore, the number of faces is $p(p-1)$. All elementary polyhedra correspond to three-line systems of which there are two for $n=6$ and five for $n=10$. These remarks are not generally true for $n=15$. Here we have six sets of five vectors ${}^2\mathbf{e}_i$ and ten sets of three vectors ${}^2\mathbf{e}_i$ forming coplanar sets. These sets are listed in Table 15. Thus faces of the $n=15$ polyhedra may be $2q$-gons ($q=2,3,4,5$) occurring in both regular and irregular form. The elementary

Table 15. *List of coplanar vectors* ${}^2\mathbf{e}_i$

1	2	7	10	15
2	3	6	8	11
3	4	7	9	12
4	5	8	10	13
1	5	6	9	14
11	12	13	14	15

1	3	13
1	4	11
2	4	14
2	5	12
3	5	15
6	7	13
6	10	12
7	8	14
8	9	15
9	10	11

polyhedra will include those corresponding to p-line systems $p=q+1$, where q vectors ${}^2\mathbf{e}_i$ are coplanar. The number of faces of these elementary polyhedra is $2(q+1)$. The expression for the number of faces of a composite $n=15$ polyhedron is more complex and depends on the number of different $2q$-gonal faces f_q, so that

$$f=\sum_{q=2}^{5} f_q.$$

These numbers satisfy the Fedorov expression

$$\binom{p}{2}=\tfrac{1}{2}\sum_{q=2}^{5}\binom{q}{2}f_q.$$

We have not presented the full list of figures for all polyhedra because of the large number. However, the selection (Figs. 2-21) includes those showing high symmetry and interesting features. These occur in particular with the larger polyhedra. The figures are drawn from a central perspective in which the vector ${}^5\mathbf{e}_6$ extends outwards perpendicular to the plane of the page. The choice of perspective emphasizes the fivefold symmetry wherever it occurs.

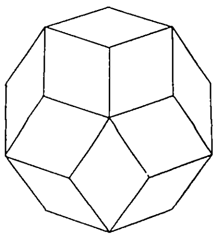

Fig. 2. (0,6), ()(1 2 3 4 5 6)†, $A(5)$. The entries, as for each of the following figures, correspond to the entries in Table 1 and are respectively $(p, n-p)$, the orbit representative of X_p^n and the stability group S_p^n. The dagger sign † designates which of the stars, *i.e.* p-star or $(n-p)$-star, is used in obtaining the polyhedron figure. The figure represents the rhombic triacontahedron. Because of the angle of perspective, five faces of the front half of the figure are hidden.

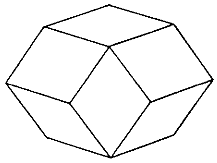

Fig. 3. (1, 5), (1) (2 3 4 5 6)†, $D(5)$.

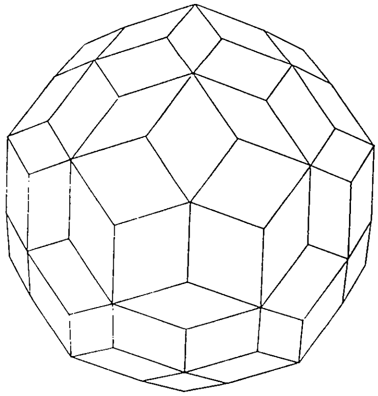

Fig. 4. (0, 10), ()(1 2 3 4 5 6 7 8 9 10)†, $A(5)$. The figure represents a rhombic enneahedron constructed from two rhombi.

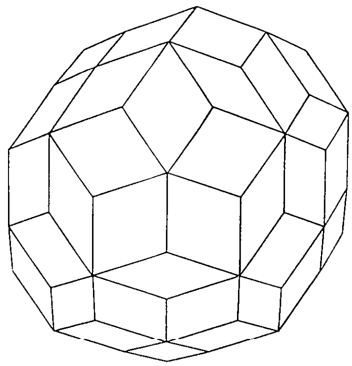

Fig. 5. (1, 9), (1) (2 3 4 5 6 7 8 9 10)†, $D(3)$.

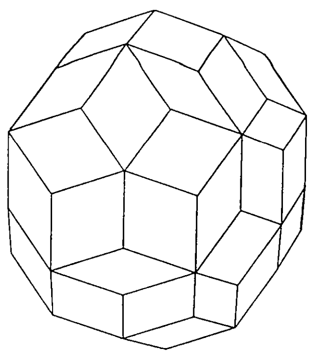

Fig. 6. (2, 8), (1 2) (3 4 5 6 7 8 9 10)†, $D(2)$.

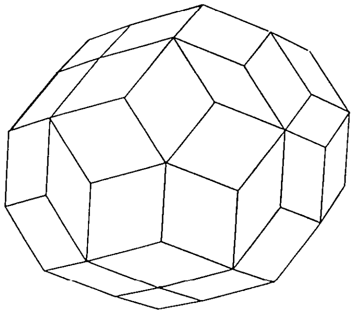

Fig. 7. (2, 8), (1 3) (2 4 5 6 7 8 9 10)†, $C(2)$.

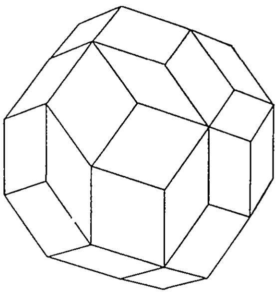

Fig. 8. (3, 7), (1 3 7) (2 4 5 6 8 9 10)†, $D(3)$.

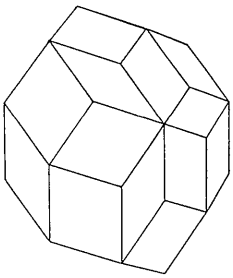

Fig. 9. (4, 6), (1 2 3 7) (4 5 6 8 9 10)†, $D(3)$.

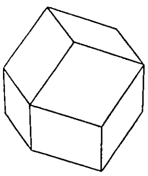

Fig. 10. (4, 6), (1 3 9 10)† (2 4 5 6 7 8), $A(4)$.

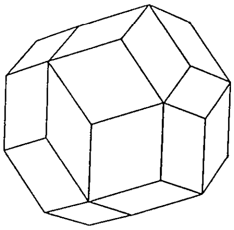

Fig. 11. (4, 6), (1 3 9 10) (2 4 5 6 7 8)†, $A(4)$.

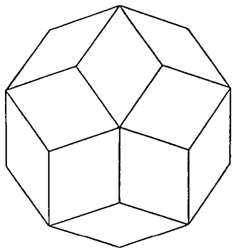

Fig. 12. (5, 5), (1 2 3 4 5) (6 7 8 9 10)†, $D(5)$.

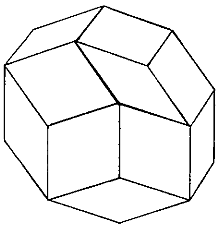

Fig. 13. (5, 5), (1 2 3 4 10) (5 6 7 8 9)†, $C(2)$.

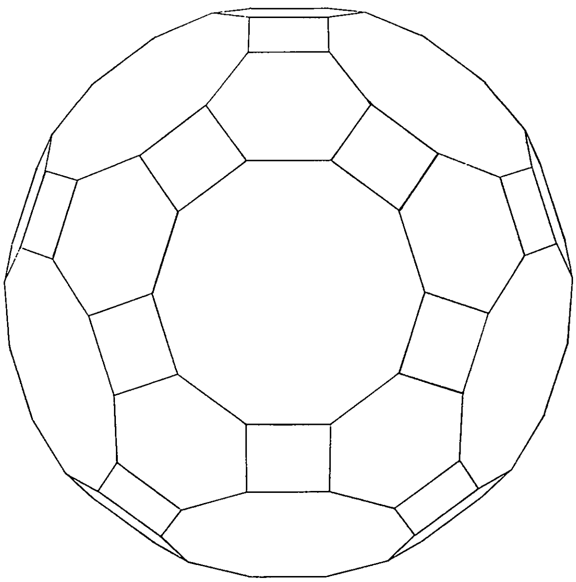

Fig. 14. (0, 15), ()(1 2 3 4 5 6 7 8 9 10)†, $A(5)$. The figure is Kepler's truncated icosidodecahedron consisting of 6 pairs of decagons, 10 pairs of octagons and 15 pairs of squares.

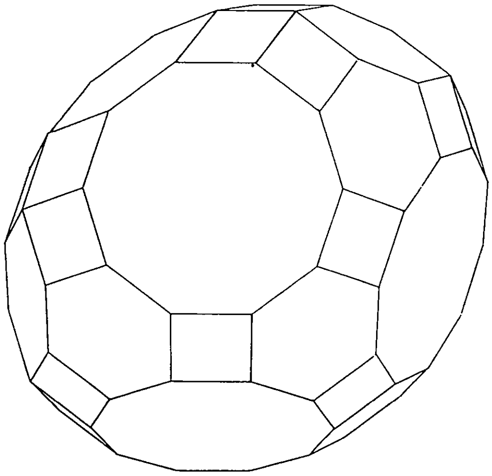

Fig. 15. (3, 12), (1 2 6) (3 4 5 7 8 9 10 11 12 13 14 15)†, *D*(3).

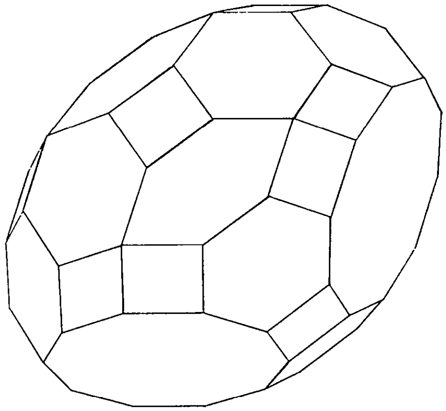

Fig. 18. (4, 11), (1 2 11 14) (3 4 5 6 7 8 9 10 12 13 15)†, *D*(2).

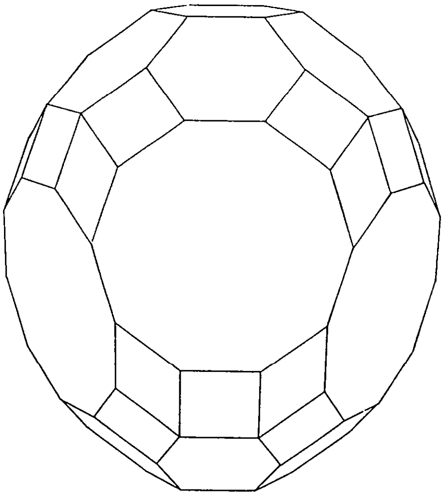

Fig. 16. (3, 12), (1 7 9) (2 3 4 5 6 8 10 11 12 13 14 15)†, *D*(3).

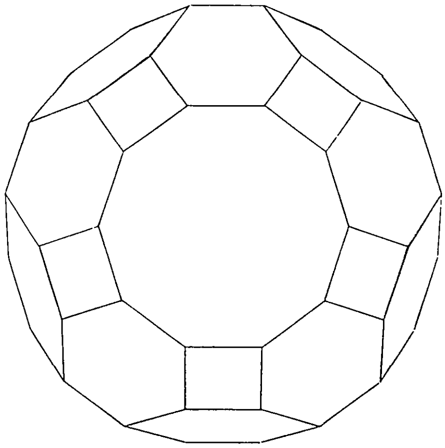

Fig. 19. (5, 10), (1 2 3 4 5) (6 7 8 9 10 11 12 13 14 15)†, *D*(5).

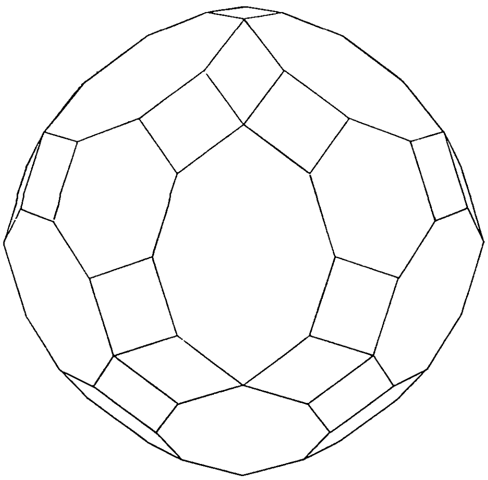

Fig. 17. (3, 12), (1 8 12) (2 3 4 5 6 7 9 10 11 13 14 15)†, *A*(4).

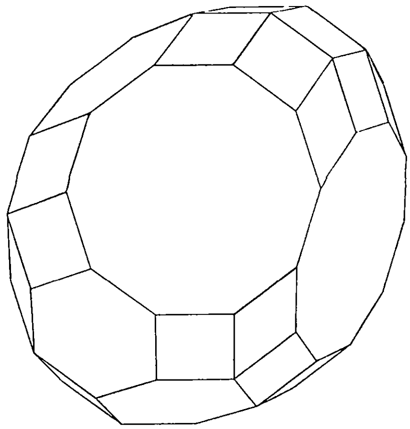

Fig. 20. (5, 10), (1 2 3 6 9) (4 5 7 8 10 11 12 13 14 15)†, *E*.

Concluding remarks

The general method described in the first part of this paper has been used in the second part to derive the cells for three quasilattices associated with the icosahedral group. For the case $n=6$ it is shown by Kramer (1986) that these cells play a role not only in the description, but also in the construction of the quasilattices. A similar role is expected for the new cells and quasilattices corresponding to $n=10$ and 15.

In response to the first referee's report, the second paragraph of the *Introduction* has been added to provide a background for some of the concepts used in the paper.

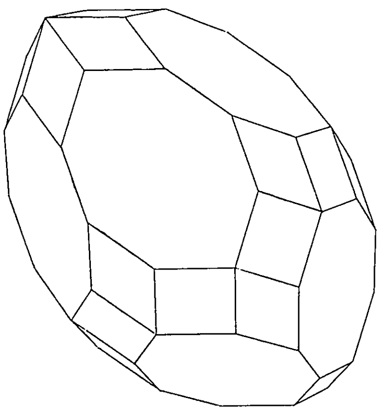

Fig. 21. (5, 10), (1 2 7 10 15) (3 4 5 6 8 9 11 12 13 14)†, $D(5)$.

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SHORT COMMUNICATIONS

Contributions intended for publication under this heading should be expressly so marked; they should not exceed about 1000 words; they should be forwarded in the usual way to the appropriate Co-editor; they will be published as speedily as possible.

Acta Cryst. (1987). **A43**, 587-588

Tables for the third-order elastic tensor in crystals. By F. G. FUMI, *Dipartimento di Fisica, Università di Genova e CISM/MPI-GNSM/CNR, Unità di Genova, Italy*

(*Received* 9 *February* 1987; *accepted* 17 *March* 1987)

Abstract

It is pointed out that the tables for the third-order elastic tensor in crystals first given by Fumi [*Phys. Rev.* (1951). **83**, 1274-1275; *Phys. Rev.* (1952). **86**, 561] have been inconsistently reported for trigonal and hexagonal groups by Huntington [*Solid State Physics* (1958). Vol. 7, pp. 213-351. New York: Academic Press] and by Mason [*Crystal Physics of Interaction Processes* (1966). New York: Academic Press], and this has caused confusion in the literature. The tables reported by Brugger [*J. Appl. Phys.* (1965). **36**, 759-768] for all crystallographic groups, which are often quoted in the more recent literature, actually coincide with those given by Fumi (1951, 1952) even in the choice of independent components.

The 'history' of the tables for the third-order elastic tensor in crystals has unfortunately been rather involved and this has created considerable confusion in the literature that has not yet been completely clarified. The tables for the

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